Killing symmetries of generalized Minkowski spaces. 1- Algebraic-infinitesimal structure of space-time rotation groups

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Abstract

In this paper, we introduce the concept of N-dimensional generalized Minkowski space, i.e. a space endowed with a (in general non-diagonal) metric tensor, whose coefficients do depend on a set of non-metrical coodinates. This is the first of a series of papers devoted to the investigation of the Killing symmetries of generalized Minkowski spaces. In particular, we discuss here the infinitesimal-algebraic structure of the space-time rotations in such spaces. It is shown that the

maximal Killing group of these spaces is the direct product of a generalized Lorentz group and a generalized translation group. We derive the explicit form of the generators of the generalized Lorentz group in the self-representation and their related, generalized Lorentz algebra. The results obtained are specialized to the case of a 4-dimensional, "deformed" Minkowski space $\widetilde{M_4}$, i.e. a pseudoeuclidean space with metric coefficients depending on energy.

1 Introduction

In the last years, two of the present authors (F.C. and R.M.) proposed a generalization of *Standard Special Relativity* (SR) based on a "deformation" of space-time, assumed to be endowed with a metric whose coefficients depend on the energy of the process considered [1]. Such a formalism (*Deformed Special Relativity*, DSR) applies in principle to *all* four interactions (electromagnetic, weak, strong and gravitational) - at least as far as their nonlocal behavior and nonpotential part is concerned - and provides a metric representation of them (at least for the process and in the energy range considered) ([1]-[5]). Moreover, it was shown that such a formalism is actually a five-dimensional one, in the sense that the deformed Minkowski space is embedded in a larger Riemannian manifold, with energy as fifth dimension [6].

In this paper, following the line of mathematical-formal research started with [7] and [8], we introduce the concept of N-dimensional generalized Minkowski space, i.e. a space endowed with a (in general non-diagonal) metric tensor, whose coefficients do depend on a set of non-metrical coodinates. The deformed space-time $\widetilde{M_4}$ of DSR is just a special case of such

spaces. This is the first of a series of papers devoted to the investigation of the Killing symmetries of generalized Minkowski spaces. In particular, we shall discuss here the infinitesimal-algebraic structure of the space-time rotations in such spaces.

The organization of the paper is as follows. In Sect. 2 we briefly review the formalism of DSR4 and of the deformed Minkowski space $\widetilde{M}_4$. Generalized Minkowski spaces are defined in Subsect. 3.1. In Subsect. 3.2 we look for the group of isometries of such spaces by means of the Killing equations. It is shown that the maximal Killing group of these spaces is the semidirect product of a generalized Lorentz group and a generalized translation group. The infinitesimal structure of the generalized Lorentz group is discussed in Sect. 4, where we derive the explicit form of its generators in the self-representation and their related, generalized Lorentz algebra. The special case of the deformed space $\widetilde{M}_4$ of DSR4 is considered in Sect. 5. Sect. 6 concludes the paper.

2 Deformed Special Relativity in four dimensions (DSR4)

The generalized ("deformed") Minkowski space $\widetilde{M}_4$ (DMS4) is defined as a space with the same local coordinates x of M_4 (the four-vectors of the usual Minkowski space), but with metric given by the metric tensor[1]

$$g_{\mu\nu,DSR4}(x^5) = diag(b_0^2(x^5), -b_1^2(x^5), -b_2^2(x^5), -b_3^2(x^5)) =$$

$$\stackrel{\text{ESC off}}{=} \delta_{\mu\nu}\left[b_0^2(x^5)\delta_{\mu 0} - b_1^2(x^5)\delta_{\mu 1} - b_2^2(x^5)\delta_{\mu 2} - b_3^2(x^5)\delta_{\mu 3}\right] \tag{1}$$

where the $\left\{b_\mu^2(x^5)\right\}$ are dimensionless, real, positive functions of the independent, non-metrical (n.m.) variable x^5 [2]. The generalized interval in $\widetilde{M}_4$ is

[1]In the following, we shall employ the notation "ESC on" ("ESC off") to mean that the Einstein sum convention on repeated indices is (is not) used.

[2]Such a coordinate is to be interpreted as the energy (see Refs. [1]-[5]); moreover, the index 5 explicitly refers to the above-mentioned fact that the deformed Minkowski space can be *"naturally" embedded* in a five-dimensional (Riemannian) space [6].

therefore given by ($x^{\mu} = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$, with c being the usual light speed in vacuum):

$$ds^2 = b_0^2 c^2 (dt)^2 - \left[b_1^2 (dx)^2 + b_2^2 (dy)^2 + b_3^2 (dz)^2\right] = g_{\mu\nu,DSR4} dx^{\mu} dx^{\nu} = dx * dx. \tag{2}$$

The last step in (2) defines the scalar product $*$ in the deformed Minkowski space $\widetilde{M_4}$. In order to emphasize the dependence of DMS4 on the variable x^5, we shall sometimes use the notation $\widetilde{M_4}(x^5)$. It follows immediately that it can be regarded as a particular case of a Riemann space with null curvature.

From the general condition

$$g_{\mu\nu,DSR4}(x^5) g^{\nu\rho}_{DSR4}(x^5) = \delta_{\mu}^{\ \rho} \tag{3}$$

we get for the contravariant components of the metric tensor

$$g^{\mu\nu}_{DSR4}(x^5) = diag(b_0^{-2}(x^5), -b_1^{-2}(x^5), -b_2^{-2}(x^5), -b_3^{-2}(x^5)) =$$

$$\overset{\text{ESC off}}{=} \delta^{\mu\nu} \left(b_0^{-2}(x^5)\delta^{\mu 0} - b_1^{-2}(x^5)\delta^{\mu 1} - b_2^{-2}(x^5)\delta^{\mu 2} - b_3^{-2}(x^5)\delta^{\mu 3}\right) \tag{4}$$

Let us stress that metric (1) is supposed to hold at a *local* (and not global) scale. We shall therefore refer to it as a "*topical*" deformed metric, because it is supposed to be valid not everywhere, but only in a suitable (local) space-time region (characteristic of both the system and the interaction considered).

The two basic postulates of DSR4 (which generalize those of standard SR) are [1]:

1- *Space-time properties:* Space-time is homogeneous, but space is not necessarily isotropic; a reference frame in which space-time is endowed with such properties is called a "*topical" reference frame* (TIRF). Two TIRF's are in general moving uniformly with respect to each other (i.e., as in SR, they are connected by a "inertiality" relation, which defines an equivalence class of ∞^3 TIRF);

2- *Generalized Principle of Relativity* (or *Principle of Metric Invariance*): All physical measurements within each TIRF must be carried out via the *same* metric.

The metric (1) is just a possible realization of the above postulates. We refer the reader to Ref.s [1]-[5] for the explicit expressions of the phenomenological energy-dependent metrics for the four fundamental interactions[3].

3 Maximal Killing group of N-d. generalized Minkowski spaces

3.1 Generalized Minkowski spaces

We shall call *generalized Minkowski space* $\widetilde{M_N}(\{x\}_{n.m.})$ a N-dimensional Riemann space with a global metric structure determined by the (in general non-diagonal) metric tensor $g_{\mu\nu}(\{x\}_{n.m.})$ $(\mu, \nu = 1, 2, ..., N)$, where $\{x\}_{n.m.}$ denotes a set of $N_{n.m.}$ non-metrical coordinates (i.e. different from the N coordinates related to the dimensions of the space considered). The metric interval in $\widetilde{M_N}(\{x\}_{n.m.})$ therefore reads (ESC on throughout)

$$ds^2 = g_{\mu\nu}(\{x\}_{n.m.})dx^\mu dx^\nu. \tag{5}$$

We shall assume the signature (T, S) (T timelike dimensions and $S = N - T$ spacelike dimensions). It follows that $\widetilde{M_N}(\{x\}_{n.m.})$ is *flat*, because *all* the components of the Riemann-Christoffel tensor vanish.

Of course, an example is just provided by the 4-d. deformed Minkowski space $\widetilde{M_4}(x^5)$.

3.2 Killing equations in a generalized Minkowski space

The Lie algebra of isometrical transformations of a N-dimensional Riemannian space is determined by the solutions of the $N(N+1)/2$ Killing equations

$$\xi_\mu(\underline{x})_{;\nu} + \xi_\nu(\underline{x})_{;\mu} = 0 \Longleftrightarrow \xi_{[\mu}(\underline{x})_{;\nu]} = 0, \tag{6}$$

[3]Since the metric coefficients $b^2_\mu(x^5)$ are *dimensionless*, they actually do depend on the ratio $\frac{x^5}{x^5_0}$, where

$$x^5_0 \equiv \ell_0 E_0$$

is a *fundamental length*, proportional (by the *dimensionally-transposing* constant ℓ_0) to the *threshold energy* E_0, characteristic of the interaction considered (see Ref.s [1]-[5]).

where the bracket [...] means symmetrization with respect to the enclosed indices, $;\mu$ denotes, as usual, covariant derivation with respect to the coordinate x^μ, and the contravariant Killing vector $\xi^\mu(\underline{x})$ is the infinitesimal vector of the general coordinate transformation

$$(x')^\mu = (x^\mu)' = x^\mu + \delta x^\mu(\underline{x}), \tag{7}$$

namely:

$$\delta x^\mu(\underline{x}) = \xi^\mu(\underline{x}). \tag{8}$$

In the case of a N-d. generalized Minkowski space $\widetilde{M_N}(\{x\}_{n.m.})$, being (as noted above) a special case of a Riemann space with constant (zero) curvature, we can state that it is a maximally symmetric space, i.e. admits a maximal Killing group with $N(N+1)/2$ parameters. Moreover, covariant derivative reduces to ordinary ones $(,\mu = \partial/\partial x^\mu)$, whence the Killing equations (6) reduce to

$$\xi_{[\mu}(\underline{x})_{,\nu]} = 0 \Longleftrightarrow \xi_\mu(\underline{x})_{,\nu} + \xi_\nu(\underline{x})_{,\mu} = 0 \Longleftrightarrow \frac{\partial \xi_\mu(\underline{x})}{\partial x^\nu} + \frac{\partial \xi_\nu(\underline{x})}{\partial x^\mu} = 0. \tag{9}$$

By virtue of the use of the omomorphic exponential mapping, any finite element g of a Lie group G_L of order M, acting on $\widetilde{M_N}(\{x\}_{n.m.})$, can be written in the exponential form

$$g = \exp\left(\sum_{i=1}^{M} \alpha_i T^i\right), \tag{10}$$

where $\{T^i\}_{i=1...M}$ is the generator basis of the Lie algebra of G_L, and $\{\alpha_i\}_{i=1...M} \in R^M$ $(\{\alpha_i\} = \{\alpha_i\}(g))$.

Therefore, by a series development of the exponential:

$$g = \exp\left(\sum_{i=1}^{M} \alpha_i T^i\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{i=1}^{M} \alpha_i(g) T^i\right)^k, \tag{11}$$

and thus we get, for an infinitesimal element $(g \to \delta g)$ $(\Leftrightarrow \{\alpha_i(g)\}_{i=1\ldots M} \in R^M \to \{\alpha_i(g)\}_{i=1\ldots M} \in I_{\underline{0}} \subset R^M$):

$$\delta g = 1 + \sum_{i=1}^{M} \alpha_i(g) T^i + O(\{\alpha_i^2(g)\}). \tag{12}$$

Then, $\forall \underline{x} \in \widetilde{M_N}(\{x\}_{n.m.})$, one has, for the action of a finite and an infinitesimal element of G_L, respectively:

$$g\underline{x} = \left[\exp\left(\sum_{i=1}^{M} \alpha_i(g) T^i\right)\right] \underline{x} = \left[\sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{i=1}^{M} \alpha_i(g) T^i\right)^k\right] \underline{x} = \underline{x'} \in \widetilde{M_N}(\{x\}_{n.m.}) \tag{13}$$

$$\left.\begin{array}{l} (\delta g)\, \underline{x} = \left[1 + \sum_{i=1}^{M} \alpha_i(g) T^i\right] \underline{x} = \underline{x} + \left(\sum_{i=1}^{M} \alpha_i(g) T^i\right) \underline{x} = \underline{x'} \in \widetilde{M_N}(\{x\}_{n.m.}) \\ \\ \delta g : \widetilde{M_N}(\{x\}_{n.m.}) \ni \underline{x} \to \underline{x'} = \underline{x} + \underline{\delta x}_{(g)}(\underline{x}) \in \widetilde{M_N}(\{x\}_{n.m.}) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \underline{\delta x}_{(g)}(\underline{x}) = \left(\sum_{i=1}^{M} \alpha_i(g) T^i\right) \underline{x} \tag{14}$$

In index notation, Eq. (14) can be written as

$$\delta x^{\mu}_{(g)}(\underline{x}) = \left[\left(\sum_{i=1}^{M} \alpha_i(g) T^i\right) \underline{x}\right]^{\mu}, \mu = 1, ..., N; \tag{15}$$

moreover, from (8) one gets

$$\xi^{\mu}_{(g)}(\underline{x}) = \left[\left(\sum_{i=1}^{M} \alpha_i(g) T^i\right) \underline{x}\right]^{\mu}. \tag{16}$$

We can now define the mixed 2-rank N-tensor $\delta\omega^{\mu}{}_{\nu}(g,\{x\}_{n.m.})$ of infinitesimal transformation (associated to $\delta g \in G_L$) as (ESC on throughout):

$$\delta x^{\mu}_{(g)}(\underline{x},\{x\}_{n.m.}) = \left[\left(\sum_{i=1}^{M}\alpha_i(g)T^i(\{x\}_{n.m.})\right)\underline{x}\right]^{\mu} \equiv \delta\omega^{\mu}{}_{\nu}(g,\{x\}_{n.m.})x^{\nu}. \tag{17}$$

The number of independent components of tensor $\delta\omega^{\mu}{}_{\nu}(g,\{x\}_{n.m.})$ is equal to the order M of the Lie group; in general, nothing can be said about its symmetry properties. From Eq.s (15)-(17) it follows that

$$\xi^{\mu}_{(g)}(\underline{x},\{x\}_{n.m.}) = \delta\omega^{\mu}{}_{\nu}(g,\{x\}_{n.m.})x^{\nu}, \tag{18}$$

which shows that $\delta\omega^{\mu}{}_{\nu}(g,\{x\}_{n.m.})$ is the tensor of the rotation parameters in $\widetilde{M_N}(\{x\}_{n.m.})$[4].

Since we are looking for the Killing groups of $\widetilde{M_N}(\{x\}_{n.m.})$ (not necessarily maximal), we have (from Eq.s (14), (15) and (17)) :

$$\xi_{(g)\mu}(\underline{x})_{,\nu} + \xi_{(g)\nu}(\underline{x})_{,\mu} = 0 \Leftrightarrow \delta x_{(g)\mu}(\underline{x})_{,\nu} + \delta x_{(g)\nu}(\underline{x})_{,\mu} = 0 \Leftrightarrow$$

$$\Leftrightarrow \left[\left(\sum_{i=1}^{M}\alpha_i(g)T^i\right)\underline{x}\right]_{\mu,\nu} + \left[\left(\sum_{i=1}^{M}\alpha_i(g)T^i\right)\underline{x}\right]_{\nu,\mu} = 0 \Leftrightarrow$$

$$\Leftrightarrow \left(\delta\omega_{\mu\rho}(g)x^{\rho}\right)_{,\nu} + \left(\delta\omega_{\nu\rho}(g)x^{\rho}\right)_{,\mu} = 0 \Leftrightarrow$$

$$\Leftrightarrow \left(\delta\omega_{[\mu\rho}(g)x^{\rho}\right)_{,\nu]} = 0. \tag{19}$$

The last equation implies *antisymmetry* of $\delta\omega_{\mu\nu}(g)$:

$$\delta\omega_{\mu\nu}(g) + \delta\omega_{\nu\mu}(g) = 0, \tag{20}$$

which therefore has $N(N-1)/2$ independent components (such a number, as stressed above, is also equal to to the order M of G_L,i.e. $M = N(N-1)/2$),

[4]In the following, for simplicity of notation, we shall often omit the explicit dependence of quantities on the non-metrical coordinates $\{x\}_{n.m.}$.

i.e. the (rotation) transformation group related to the tensor $\delta\omega_{\mu\nu}(g)$ is a $N(N-1)/2$-parameter Killing group.

Since a N-d. generalized Minkowski space is maximally symmetric, we have still to find another N-parameter Killing group of $\widetilde{M_N}(\{x\}_{n.m.})$ (because $N + N(N-1)/2 = N(N+1)/2$).

This is easily done by noting that the $N(N+1)/2$ Killing equations (9) in such a space are trivially satisfied by constant covariant N-vectors $\xi_\mu \neq \xi_\mu(\underline{x})$, to which there corresponds the infinitesimal transformation

$$\delta g : x^\mu \to (x')^\mu (\underline{x}, \{x\}_{n.m.}) = (x^\mu)' (\underline{x}, \{x\}_{n.m.}) =$$

$$= x^\mu + \delta x^\mu_{(g)}(\{x\}_{n.m.}) = x^\mu + \xi^\mu_{(g)}(\{x\}_{n.m.}), \tag{21}$$

with $\delta x^\mu_{(g)}(\{x\}_{n.m.})$ and $\xi^\mu_{(g)}(\{x\}_{n.m.})$ constant (with respect to x^μ).

In conclusion, a N-d. generalized Minkowski space $\widetilde{M_N}(\{x\}_{n.m.})$ admits a maximal Killing group which is the (semidirect) product of the Lie group of N-dimensional space-time rotations (or N-d. generalized, homogeneous Lorentz group $SO(T,S)^{N(N-1)/2}_{GEN.}$) with $N(N-1)/2$ parameters, and of the Lie group of N-dimensional space-time translations $Tr.(T,S)^N_{GEN.}$ with N parameters :

$$P(T,S)^{N(N+1)/2}_{GEN.} = SO(T,S)^{N(N-1)/2}_{GEN.} \otimes_s Tr.(T,S)^N_{GEN.}. \tag{22}$$

We will refer to it as the *generalized Poincaré* (or *inhomogeneous Lorentz*) *group* $P(S,T)^{N(N+1)/2}_{GEN.}$.

3.2.1 Solving the Killing equations in a 4-d. generalized Minkowski space

We want now to find the explicit solutions of the Killing equations in a 4-d. generalized Minkowski space $\widetilde{M_4}(\{x\}_{n.m.})$ $(S \leq 4, T = 4 - S)$. A covariant

Killing 4-vector $\xi_\mu(x^0, x^1, x^2, x^3)$ must satisfy Eq. (9), namely

$$\left\{\begin{array}{l}
I. \quad \dfrac{\partial\xi_0(\{x\}_{m.})}{\partial x^0} = 0 \\[3ex]
II. \quad \dfrac{\partial\xi_0(\{x\}_{m.})}{\partial x^1} + \dfrac{\partial\xi_1(\{x\}_{m.})}{\partial x^0} = 0 \\[3ex]
III. \quad \dfrac{\partial\xi_0(\{x\}_{m.})}{\partial x^2} + \dfrac{\partial\xi_2(\{x\}_{m.})}{\partial x^0} = 0 \\[3ex]
IV. \quad \dfrac{\partial\xi_0(\{x\}_{m.})}{\partial x^3} + \dfrac{\partial\xi_3(\{x\}_{m.})}{\partial x^0} = 0 \\[3ex]
V. \quad \dfrac{\partial\xi_1(\{x\}_{m.})}{\partial x^1} = 0 \\[3ex]
VI. \quad \dfrac{\partial\xi_1(\{x\}_{m.})}{\partial x^2} + \dfrac{\partial\xi_2(\{x\}_{m.})}{\partial x^1} = 0 \\[3ex]
VII. \quad \dfrac{\partial\xi_1(\{x\}_{m.})}{\partial x^3} + \dfrac{\partial\xi_3(\{x\}_{m.})}{\partial x^1} = 0 \\[3ex]
VIII. \quad \dfrac{\partial\xi_2(\{x\}_{m.})}{\partial x^2} = 0 \\[3ex]
IX. \quad \dfrac{\partial\xi_2(\{x\}_{m.})}{\partial x^3} + \dfrac{\partial\xi_3(\{x\}_{m.})}{\partial x^2} = 0 \\[3ex]
X. \quad \dfrac{\partial\xi_3(\{x\}_{m.})}{\partial x^3} = 0
\end{array}\right. \tag{23}$$

From Eq.s $I.$,$V.$,$VIII.$ and $X.$ one trivially gets:

$$\begin{cases} \xi_0 = \xi_0(x^1, x^2, x^3) \\ \\ \xi_1 = \xi_1(x^0, x^2, x^3) \\ \\ \xi_2 = \xi_2(x^0, x^1, x^3) \\ \\ \xi_3 = \xi_3(x^0, x^1, x^2) \end{cases} . \tag{24}$$

In general, solving the system (23) of 10 coupled partial differential equations (PDEs) in 4 functional unknowns $(\xi_0, \xi_1, \xi_2, \xi_3)$ of 4 independent variables $(\{x\}_{m.}) = (x^0, x^1, x^2, x^3)$ is cumbersome, but straightforward. The final result is:

$$\begin{cases} \xi_0(\{x\}_{m.}) = -\zeta^1 x^1 - \zeta^2 x^2 - \zeta^3 x^3 + T^0 \\ \\ \xi_1(\{x\}_{m.}) = \zeta^1 x^0 + \theta^2 x^3 - \theta^3 x^2 - T^1 \\ \\ \xi_2(\{x\}_{m.}) = \zeta^2 x^0 - \theta^1 x^3 + \theta^3 x^1 - T^2 \\ \\ \xi_3(\{x\}_{m.}) = \zeta^3 x^0 + \theta^1 x^2 - \theta^2 x^1 - T^3 \end{cases} \tag{25}$$

where ζ^i, θ^i $(i = 1, 2, 3)$ and T^μ $(\mu = 0, 1, 2, 3)$ are real coefficients.

Thus, we can draw the following conclusions:

1- In spite of the fact that no assumption was made on the functional form of the Killing vector, we got a dependence at most linear (inhomogeneous) on metric coordinates for all components of $\xi_\mu(\{x\}_{m.})$. Therefore, in order to determine the (maximal) Killing group of a generalized Minkowski space [5], one can, without loss of generality, consider only groups whose transformation representation is implemented by transformations at most linear in the coordinates.

[5]Indeed, although we discussed explicitly the 4-d. case, the extension to the generic N-d. case is straightforward.

2- In general, $\xi_\mu \neq \xi_\mu(\{x\}_{n.m.})$, i.e. the covariant Killing vector does not depend on possible non-metric variables. On the contrary, the *contravariant* Killing 4-vector *does indeed*, due to dependence of the fully contravariant metric tensor on $\{x\}_{n.m.}$:

$$\xi^\mu(\{x\}_{m.}, \{x\}_{n.m.}) = g^{\mu\nu}(\{x\}_{n.m.})\xi_\nu(\{x\}_{m.}). \tag{26}$$

Such a result is consistent with the fact that $\delta\omega_{\mu\nu}(g)$, unlike $\delta\omega^\mu{}_\nu(g, \{x\}_{n.m.})$, is independent of $\{x\}_{n.m.}$ (cfr. Eq.s (18) and (19)).

3 - Solution (25) *does not depend on the metric tensor.* This implies that all 4-d. generalized Minkowski spaces admit the same *covariant* Killing 4-vector. *It corresponds to the covariant 4-vector of infinitesimal transformation of the space-time roto-translational group of* $\widetilde{M_4}(\{x\}_{n.m.})$. Therefore, by (25) and (26), and assuming the signature hyperbolic $(+, -, -, -)$ (i.e. $S = 3$, $T = 1$), in a basis of "length-dimensional" coordinates, we can state that:

3.a) $\underline{\zeta} = (\zeta^1, \zeta^2, \zeta^3)$ is the 3-vector of dimensionless parameters ("rapidity") of generalized 3-d. boost;

3.b) $\underline{\theta} = (\theta^1, \theta^2, \theta^3)$ is the 3-vector of dimensionless parameters (angles) of generalized 3-d. rotation;

3.c) $T_\mu = (T^0, -T^1, -T^2, -T^3)$ is the covariant 4-vector of ("length-dimensional") parameters of generalized 4-d. translation.

4 Infinitesimal structure of generalized space-time rotation groups

4.1 Finite-dimensional representation of infinitesimal generators and generalized Lorentz algebra

As in the standard special-relatistic case, we can decompose the mixed N-tensor of infinitesimal transformation parameters $\delta\omega^\mu{}_\nu(g, \{x\}_{n.m.})$ (see (17)) as [6]:

$$\delta\omega^\mu{}_\nu(g, \{x\}_{n.m.}) = \frac{1}{2}\delta\omega_{\alpha\beta}(g)(I^{\alpha\beta})^\mu{}_\nu(\{x\}_{n.m.}) \tag{27}$$

[6]The factor $\frac{1}{2}$ is inserted only for further convenience.

i.e. as a linear combination of $N(N-1)/2$ matrices (indipendent of the group element g) $\left\{(I^{\alpha\beta})^{\mu}{}_{\nu}(\{x\}_{n.m.})\right\}_{\alpha,\beta=1...N}$ [7] with coefficients $\{\delta\omega_{\alpha\beta}(g)\}_{\alpha,\beta=1...N}$. Such matrices represent *the infinitesimal generators* of the space-time rotational component of the maximal Killing group of $\widetilde{M_N}(\{x\}_{n.m.})$. Since in this case $\delta\omega_{\mu\nu}(g)$ is antisymmetric (see (20)), also the basis matrices $\left\{(I^{\alpha\beta})^{\mu}{}_{\nu}(\{x\}_{n.m.})\right\}_{\alpha,\beta=1...N}$, are antisymmetric in indices α and β :

$$\left\{(I^{\alpha\beta})^{\mu}{}_{\nu}(\{x\}_{n.m.})\right\}_{\alpha,\beta=1...N} = -\left\{(I^{\beta\alpha})^{\mu}{}_{\nu}(\{x\}_{n.m.})\right\}_{\alpha,\beta=1...N}. \tag{28}$$

For the fully covariant $\delta\omega_{\mu\nu}(g)$ the analogous decomposition reads

$$\delta\omega_{\mu\nu}\ (g) = g_{\mu\rho}(\{x\}_{n.m.})\delta\omega^{\rho}{}_{\nu}(g,\{x\}_{n.m.}) =$$

$$= \frac{1}{2}\delta\omega_{\alpha\beta}(g)g_{\mu\rho}(\{x\}_{n.m.})(I^{\alpha\beta})^{\rho}{}_{\nu}(\{x\}_{n.m.}) =$$

$$= \frac{1}{2}\delta\omega_{\alpha\beta}(g)(I^{\alpha\beta})_{\mu\nu}(\{x\}_{n.m.}). \tag{29}$$

But, since $\delta\omega_{\mu\nu}(g)$ is independent of $\{x\}_{n.m.}$, the same holds for its components $(I^{\alpha\beta})_{\mu\nu}$, and therefore Eq. (29) implies

$$(I^{\alpha\beta})_{\mu\nu} \neq (I^{\alpha\beta})_{\mu\nu}(\{x\}_{n.m.}). \tag{30}$$

In order to find the explicit form of the infinitesimal generators in the N-d. matrix representation, let us exploit the antisymmetry of $\delta\omega_{\mu\nu}\ (g)$:

[7] The pair of indices (α,β) labels the different infinitesimal group generators, whereas - in the ($N(<\infty)$-dimensional) matrix representation of the generators we are considering - the controvariant (covariant) index is a row (column) index. This latter remark holds true for $\delta\omega^{\mu}{}_{\nu}(g,\{x\}_{n.m.})$, too.

$$\delta\omega_{\mu\nu}\ (g) = -\delta\omega_{\nu\mu}\ (g) \Leftrightarrow \delta\omega_{\mu\nu}\ (g) =$$

$$= \frac{1}{2}(\delta\omega_{\mu\nu}\ + \delta\omega_{\mu\nu}\ ) = \frac{1}{2}(\delta\omega_{\mu\nu}\ - \delta\omega_{\nu\mu}\ ) =$$

$$= \frac{1}{2}g^{\alpha}{}_{\mu}g^{\beta}{}_{\nu}\delta\omega_{\alpha\beta} - \frac{1}{2}g^{\beta}{}_{\mu}g^{\alpha}{}_{\nu}\delta\omega_{\alpha\beta} = \frac{1}{2}\delta\omega_{\alpha\beta}(g^{\alpha}{}_{\mu}g^{\beta}{}_{\nu} - g^{\beta}{}_{\mu}g^{\alpha}{}_{\nu}) =$$

$$= \frac{1}{2}\delta\omega_{\alpha\beta}(g)(\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu} - \delta^{\beta}{}_{\mu}\delta^{\alpha}{}_{\nu}). \tag{31}$$

Comparing (29) and (31) yields [8] :

$$g_{\mu\rho}(\{x\}_{n.m.})(I^{\alpha\beta})^{\rho}{}_{\nu}(\{x\}_{n.m.}) \equiv (I^{\alpha\beta})_{\mu\nu} = (\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu} - \delta^{\beta}{}_{\mu}\delta^{\alpha}{}_{\nu}); \tag{32}$$

we get therefore the following explicit form for the mixed matrix representa-

[8]Eq. (32) clearly shows that the factors with a non-metric dependence in $g_{\mu\rho}(\{x\}_{n.m.})$ e $(I^{\alpha\beta})^{\rho}{}_{\nu}(\{x\}_{n.m.})$ annul each other. The same *does not* happen when both μ and ν are controvariant:

$$(I^{\alpha\beta})^{\mu\nu} := g^{\mu\rho}(\{x\}_{n.m.})g^{\nu\sigma}(\{x\}_{n.m.})(I^{\alpha\beta})_{\rho\sigma} =$$

$$= g^{\mu\rho}(\{x\}_{n.m.})g^{\nu\sigma}(\{x\}_{n.m.})(\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma} - \delta^{\beta}_{\rho}\delta^{\alpha}_{\sigma}) =$$

$$= g^{\mu\alpha}(\{x\}_{n.m.})g^{\nu\beta}(\{x\}_{n.m.}) - g^{\mu\beta}(\{x\}_{n.m.})g^{\nu\alpha}(\{x\}_{n.m.}).$$

tion of the generators [9]:

$$(I^{\alpha\beta})^{\mu}{}_{\nu}(\{x\}_{n.m.}) = g^{\mu\rho}(\{x\}_{n.m.})(I^{\alpha\beta})_{\rho\nu} =$$

$$= g^{\mu\rho}(\{x\}_{n.m.})(\delta^{\alpha}{}_{\rho}\delta^{\beta}{}_{\nu} - \delta^{\beta}{}_{\rho}\delta^{\alpha}{}_{\nu}) = g^{\mu\alpha}(\{x\}_{n.m.})\delta^{\beta}{}_{\nu} - g^{\mu\beta}(\{x\}_{n.m.})\delta^{\alpha}{}_{\nu}. \tag{33}$$

It is easy to see that the generators $\left\{(I^{\alpha\beta})^{\mu}{}_{\nu}(\{x\}_{n.m.})\right\}_{\alpha,\beta=1\ldots N}$ satisfy the following Lie algebra (with the usual commutatorial implementation of Lie algebra product):

$$[I^{\alpha\beta}(\{x\}_{n.m.}), I^{\rho\sigma}(\{x\}_{n.m.})] =$$

$$= g^{\alpha\sigma}(\{x\}_{n.m.})I^{\beta\rho}(\{x\}_{n.m.}) + g^{\beta\rho}(\{x\}_{n.m.})I^{\alpha\sigma}(\{x\}_{n.m.}) +$$

$$-g^{\alpha\rho}(\{x\}_{n.m.})I^{\beta\sigma}(\{x\}_{n.m.}) - g^{\beta\sigma}(\{x\}_{n.m.})I^{\alpha\rho}(\{x\}_{n.m.}). \tag{34}$$

Eq. (34) defines the *generalized Lorentz algebra*, associated to the generalized, homogeneous Lorentz group $SO(T,S)^{N(N-1)/2}_{GEN.}$ of the N-d. generalized Minkowski space $\widetilde{M_N}(\{x\}_{n.m.})$.

4.2 The case of a 4-dimensional generalized Minkowski space

4.2.1 Self-representation of the infinitesimal generators

Let us specialize the results of the previous Subsection to a 4-d. generalized Minkowski space. Assuming therefore that Greek indices range in $\{0,1,2,3\}$,

[9] We have analogously

$$(I^{\alpha\beta})_{\mu}{}^{\nu}(\{x\}_{n.m.}) = g^{\nu\rho}(\{x\}_{n.m.})(I^{\alpha\beta})_{\mu\rho} =$$

$$= g^{\nu\rho}(\{x\}_{n.m.})(\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\rho} - \delta^{\beta}{}_{\mu}\delta^{\alpha}{}_{\rho}) =$$

$$= g^{\nu\beta}(\{x\}_{n.m.})\delta^{\alpha}_{\mu} - g^{\nu\alpha}(\{x\}_{n.m.})\delta^{\beta}{}_{\mu}.$$

and that a (not necessarily hyperbolic) signature ($S \leqslant 4, T = 4 - S$) holds, we can write explicitly the generator $I^{\alpha\beta}(\{x\}_{n.m.})$ of $SO(S, T = 4 - S)_{GEN}$ as the antisymmetric matrix:

$$I^{\alpha\beta}(\{x\}_{n.m.}) =$$

$$= \left(\begin{array}{cccc} 0 & I^{01}(\{x\}_{n.m.}) & I^{02}(\{x\}_{n.m.}) & I^{03}(\{x\}_{n.m.}) \\ -I^{01}(\{x\}_{n.m.}) & 0 & I^{12}(\{x\}_{n.m.}) & I^{13}(\{x\}_{n.m.}) \\ -I^{02}(\{x\}_{n.m.}) & -I^{12}(\{x\}_{n.m.}) & 0 & I^{23}(\{x\}_{n.m.}) \\ -I^{03}(\{x\}_{n.m.}) & -I^{13}(\{x\}_{n.m.}) & -I^{23}(\{x\}_{n.m.}) & 0 \end{array} \right). \tag{35}$$

Like any rank-2, antisymmetric 4-tensor, $I^{\alpha\beta}(\{x\}_{n.m.})$ can be expressed in terms of an axial and a polar 3-vector. By introducing the following infinitesimal generators ($i, j, k = 1, 2, 3$, ESC on throughout)

$$S^i(\{x\}_{n.m.}) \equiv \frac{1}{2}\epsilon^i{}_{jk} I^{jk}(\{x\}_{n.m.}) \tag{36}$$

$$K^i(\{x\}_{n.m.}) \equiv I^{0i}(\{x\}_{n.m.}) \tag{37}$$

(where ϵ_{ijk} is the rank-3, fully antisymmetric Levi-Civita 3-tensor, with the convention $\epsilon_{123} \equiv 1$), corresponding to the components of the axial operatorial 3-vector

$$\underline{S}(\{x\}_{n.m.}) \equiv (I^{23}(\{x\}_{n.m.}), I^{31}(\{x\}_{n.m.}), I^{12}(\{x\}_{n.m.})) \tag{38}$$

and of the polar operatorial one

$$\underline{K}(\{x\}_{n.m.}) \equiv (I^{01}(\{x\}_{n.m.}), I^{02}(\{x\}_{n.m.}), I^{03}(\{x\}_{n.m.})), \tag{39}$$

$I^{\alpha\beta}(\{x\}_{n.m.})$ can thus be rewritten as:

$$I^{\alpha\beta}(\{x\}_{n.m.}) =$$

$$= \left(\begin{array}{cccc} 0 & K^1(\{x\}_{n.m.}) & K^2(\{x\}_{n.m.}) & K^3(\{x\}_{n.m.}) \\ -K^1(\{x\}_{n.m.}) & 0 & S^3(\{x\}_{n.m.}) & -S^2(\{x\}_{n.m.}) \\ -K^2(\{x\}_{n.m.}) & -S^3(\{x\}_{n.m.}) & 0 & S^1(\{x\}_{n.m.}) \\ -K^3(\{x\}_{n.m.}) & S^2(\{x\}_{n.m.}) & -S^1(\{x\}_{n.m.}) & 0 \end{array} \right). \tag{40}$$

The set of generators $\underline{S}(\{x\}_{n.m.})$, $\underline{K}(\{x\}_{n.m.})$ constitute the *self-representation basis* for $SO(S,T=4-S)_{GEN.}$. Unlike the case of standard SR - where $\underline{S}$, $\underline{K}$ do represent the rotation and boost generators, respectively -, one cannot give them a precise physical meaning, because this latter depends on both the number S of spacelike dimensions and the assignement of dimensional labelling (for full generality, here we left it unspecified, even if it is clear that the most directly physically meaningful case is the hyperbolically signed-one: $S=3$).

4.2.2 Decomposition of the parametric 4-tensor $\delta\omega_{\mu\nu}(g)$

We can now exploit the self-representation form of the infinitesimal generators of $SO(S,T=4-S)_{GEN.}$ ($S \leq 4$) to decompose the infinitesimal parametric 4-tensor $\delta\omega_{\mu\nu}(g)$.

Eq. (17), on account of (27), can be written as

$$\delta x^{\mu}_{(g)}(\{x\}_{m.}, \{x\}_{n.m.}) = \delta\omega^{\mu}{}_{\nu}(g,\{x\}_{n.m.})x^{\nu} = \frac{1}{2}\delta\omega_{\alpha\beta}(g)(I^{\alpha\beta})^{\mu}{}_{\nu}(\{x\}_{n.m.})x^{\nu}, \tag{41}$$

which is valid in the general case of $SO(S,T=N-S)_{GEN.}$.

In the case $N=4$, we have (by (37)):

$$\delta x^{\mu}_{(g)}(\{x\}_{m.}, \{x\}_{n.m.}) = \delta\omega^{\mu}{}_{\nu}(g,\{x\}_{n.m.})x^{\nu} = \frac{1}{2}\delta\omega_{\alpha\beta}(g)(I^{\alpha\beta})^{\mu}{}_{\nu}(\{x\}_{n.m.})x^{\nu} =$$

$$= \frac{1}{2}\delta\omega_{ij}(g)(I^{ij})^{\mu}{}_{\nu}(\{x\}_{n.m.})x^{\nu} + \delta\omega_{0i}(g)(I^{0i})^{\mu}{}_{\nu}(\{x\}_{n.m.})x^{\nu} =$$

$$= \frac{1}{2}\delta\omega_{ij}(g)(I^{ij})^{\mu}{}_{\nu}(\{x\}_{n.m.})x^{\nu} + \delta\omega_{0i}(g)(K^{i})^{\mu}{}_{\nu}(\{x\}_{n.m.})x^{\nu}. \tag{42}$$

Moreover, from (36) we get:

$$S^{i}(\{x\}_{n.m.}) \equiv \frac{1}{2}\epsilon^{i}{}_{jk}I^{jk}(\{x\}_{n.m.}) \Leftrightarrow I^{jk}(\{x\}_{n.m.}) = \epsilon^{jk}_{\ \ l}S^{l}(\{x\}_{n.m.}); \tag{43}$$

replacing such result in (42) one thus has:

$$\delta x^{\mu}_{(g)}(\{x\}_{m.},\{x\}_{n.m.}) = \frac{1}{2}\delta\omega_{ij}(g)(I^{ij})^{\mu}{}_{\nu}(\{x\}_{n.m.})x^{\nu} + \delta\omega_{0i}(g)(K^{i})^{\mu}{}_{\nu}(\{x\}_{n.m.})x^{\nu} =$$

$$= \frac{1}{2}\delta\omega_{ij}(g)(\epsilon^{ij}{}_{l}S^{l}(\{x\}_{n.m.}))^{\mu}{}_{\nu}x^{\nu} + \delta\omega_{0i}(g)(K^{i})^{\mu}{}_{\nu}(\{x\}_{n.m.})x^{\nu} =$$

$$= \frac{1}{2}\epsilon^{ij}{}_{l}\delta\omega_{ij}(g)(S^{l})^{\mu}{}_{\nu}(\{x\}_{n.m.})x^{\nu} + \delta\omega_{0i}(g)(K^{i})^{\mu}{}_{\nu}(\{x\}_{n.m.})x^{\nu}. \qquad (44)$$

We can now define an axial and a polar parametric 3-vector in the following way:

$$\theta_i(g) \equiv -\frac{1}{2}\epsilon_i{}^{jk}\delta\omega_{jk}(g) \qquad (45)$$

$$\zeta_i(g) \equiv -\delta\omega_{0i}(g), \qquad (46)$$

namely

$$\underline{\theta}(g) \equiv \ (-\delta\omega_{23}(g), -\delta\omega_{31}(g), -\delta\omega_{12}(g)) \qquad (47)$$

$$\underline{\zeta}(g) \equiv (-\delta\omega_{01}(g), -\delta\omega_{02}(g), -\delta\omega_{03}(g)). \qquad (48)$$

Therefore, $\delta\omega_{\alpha\beta}(g)$ can be written in matrix form as:

$$\delta\omega_{\alpha\beta}(g) = \begin{pmatrix} 0 & -\zeta^1(g) & -\zeta^2(g) & -\zeta^3(g) \\ \zeta^1(g) & 0 & -\theta^3(g) & \theta^2(g) \\ \zeta^2(g) & \theta^3(g) & 0 & -\theta^1(g) \\ \zeta^3(g) & -\theta^2(g) & \theta^1(g) & 0 \end{pmatrix}. \qquad (49)$$

Having left the number S of spacelike dimensions and the dimensional labelling unspecified, we cannot attribute a physical meaning to the parametric 3-vectors (47),(48) (unlike the case of standard SR, where $\underline{\theta}(g)$ and $\underline{\zeta}(g)$ are the space rotation and boost parameters, respectively).

Eq. (44) can therefore be rewritten in terms of the 3-d. Euclidean scalar product $\cdot$ as:

$$\delta x^{\mu}_{(g)}(\{x\}_{m.},\{x\}_{n.m.}) = -\theta_l(g)(S^l)^{\mu}{}_{\nu}(\{x\}_{n.m.})x^{\nu} - \zeta_i(g)(K^i)^{\mu}{}_{\nu}\{x\}_{n.m.})x^{\nu} =$$

$$= \left[-\underline{\theta(g)}\cdot\underline{S}(\{x\}_{n.m.}) - \underline{\zeta(g)}\cdot\underline{K}(\{x\}_{n.m.})\right]^{\mu}{}_{\nu} x^{\nu} \tag{50}$$

5 Space-time rotation component of the Killing group in a 4-d. deformed Minkowski space

5.1 Deformed homogeneous Lorentz group $SO(3,1)_{DEF.}$ and self-representation basis of infinitesimal generators

We want now to specialize the results obtained to the case of DSR4, i.e. considering a 4-d. deformed Minkowski space $\widetilde{M}_4(x^5)$.

Let us recall that the N-dimensional representation of the infinitesimal generators of the Killing group in a N-d. generalized Minkowski space is determined (by means of eq. (33)) by the mere knowledge of its metric tensor. In the DSR4 case we have therefore:

$$(I^{\alpha\beta})^{\mu}{}_{\nu,DSR4}(x^5) = g^{\mu\rho}_{DSR4}(x^5)(I^{\alpha\beta})_{\rho\nu,DSR4} =$$

$$= g^{\mu\rho}_{DSR4}(x^5)(\delta^{\alpha}{}_{\rho}\delta^{\beta}{}_{\nu} - \delta^{\beta}{}_{\rho}\delta^{\alpha}{}_{\nu}) = g^{\mu\alpha}_{DSR4}(x^5)\delta^{\beta}{}_{\nu} - g^{\mu\beta}_{DSR4}(x^5)\delta^{\alpha}{}_{\nu} =$$

$$\stackrel{\text{ESC off}}{=} \delta^{\mu\alpha}\left(b_0^{-2}(x^5)\delta^{\mu0} - b_1^{-2}(x^5)\delta^{\mu1} - b_2^{-2}(x^5)\delta^{\mu2} - b_3^{-2}(x^5)\delta^{\mu3}\right)\delta^{\beta}{}_{\nu} +$$

$$-\delta^{\mu\beta}\left(b_0^{-2}(x^5)\delta^{\mu0} - b_1^{-2}(x^5)\delta^{\mu1} - b_2^{-2}(x^5)\delta^{\mu2} - b_3^{-2}(x^5)\delta^{\mu3}\right)\delta^{\alpha}{}_{\nu}. \tag{51}$$

From such result, we thence get the following 4×4 matrix representation of the infinitesimal generator of the deformed homogeneous Lorentz group

$SO(3,1)^6_{DEF.}$ (space-time rotation component of the deformed Poincaré group $P(3,1)^{10}_{DEF.}$):

$$I^{10}_{DSR4}(x^5) = \begin{pmatrix} 0 & -b_0^{-2}(x^5) & 0 & 0 \\ -b_1^{-2}(x^5) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{52}$$

$$I^{20}_{DSR4}(x^5) = \begin{pmatrix} 0 & 0 & -b_0^{-2}(x^5) & 0 \\ 0 & 0 & 0 & 0 \\ -b_2^{-2}(x^5) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{53}$$

$$I^{30}_{DSR4}(x^5) = \begin{pmatrix} 0 & 0 & 0 & -b_0^{-2}(x^5) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -b_3^{-2}(x^5) & 0 & 0 & 0 \end{pmatrix}, \tag{54}$$

$$I^{12}_{DSR4}(x^5) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -b_1^{-2}(x^5) & 0 \\ 0 & b_2^{-2}(x^5) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{55}$$

$$I^{23}_{DSR4}(x^5) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_2^{-2}(x^5) \\ 0 & 0 & b_3^{-2}(x^5) & 0 \end{pmatrix}, \tag{56}$$

$$I^{31}_{DSR4}(x^5) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_1^{-2}(x^5) \\ 0 & 0 & 0 & 0 \\ 0 & -b_3^{-2}(x^5) & 0 & 0 \end{pmatrix}. \tag{57}$$

A comparison of Eqs. (52)-(57) with the 4-d. matrix representation of the infinitesimal generators of the standard homogeneous Lorentz group $SO(3,1)$

shows that *the deformation of the metric structure implies the loss of symmetry of the boost generators and of antisymmetry of space-rotation generators.*

The antisymmetry of the generators in the labelling indices (α, β) still holds:

$$\left\{ (I^{\alpha\beta})^{\mu}{}_{\nu, DSR4}(x^5) \right\}_{\alpha,\beta=0,1,2,3} = - \left\{ (I^{\beta\alpha})^{\mu}{}_{\nu, DSR4}(x^5) \right\}_{\alpha,\beta=0,1,2,3}, \tag{58}$$

or, equivalently:

$$I^{\alpha\beta}_{DSR4}(x^5) = -I^{\beta\alpha}_{DSR4}(x^5), \quad \alpha, \beta = 0, 1, 2, 3. \tag{59}$$

Therefore, there are only 6 independent generators. In matrix form we get:

$$I^{\alpha\beta}_{DSR4}(x^5) = \begin{pmatrix} 0 & I^{01}_{DSR4}(x^5) & I^{02}_{DSR4}(x^5) & I^{03}_{DSR4}(x^5) \\ -I^{01}_{DSR4}(x^5) & 0 & I^{12}_{DSR4}(x^5) & I^{13}_{DSR4}(x^5) \\ -I^{02}_{DSR4}(x^5) & -I^{12}_{DSR4}(x^5) & 0 & I^{23}_{DSR4}(x^5) \\ -I^{03}_{DSR4}(x^5) & -I^{13}_{DSR4}(x^5) & -I^{23}_{DSR4}(x^5) & 0 \end{pmatrix}. \tag{60}$$

We can now pass to the self-representation basis of the generators of $SO(3,1)_{DEF.}$ by introducing the following axial and polar 3-vectors by means of the Levi-Civita tensor:

$$S^i_{DSR4}(x^5) \equiv \frac{1}{2}\epsilon^i{}_{jk} I^{jk}_{DSR4}(x^5), \tag{61}$$

$$K^i_{DSR4}(x^5) \equiv I^{0i}_{DSR4}(x^5), \tag{62}$$

or

$$\underline{S_{DSR4}}(x^5) \equiv (I^{23}_{DSR4}(x^5), I^{31}_{DSR4}(x^5), I^{12}_{DSR4}(x^5)), \tag{63}$$

$$\underline{K_{DSR4}}(x^5) \equiv (I^{01}_{DSR4}(x^5), I^{02}_{DSR4}(x^5), I^{03}_{DSR4}(x^5)). \tag{64}$$

Then, $I^{\alpha\beta}_{DSR4}(x^5)$ can be written as:

$$I^{\alpha\beta}_{DSR4}(x^5) = \begin{pmatrix} 0 & K^1_{DSR4}(x^5) & K^2_{DSR4}(x^5) & K^3_{DSR4}(x^5) \\ -K^1_{DSR4}(x^5) & 0 & S^3_{DSR4}(x^5) & -S^2_{DSR4}(x^5) \\ -K^2_{DSR4}(x^5) & -S^3_{DSR4}(x^5) & 0 & S^1_{DSR4}(x^5) \\ -K^3_{DSR4}(x^5) & S^2_{DSR4}(x^5) & -S^1_{DSR4}(x^5) & 0 \end{pmatrix}. \tag{65}$$

Due to the hyperbolicity of the assumed metric signature, in the DSR4, like in the SR case, we can identify (apart from a sign) $S^i_{DSR4}(x^5)$ with the infinitesimal generator of the deformed 3-d. space rotation around $\widehat{x^i}$, and $K^i_{DSR4}(x^5)$ with the infinitesimal generator of the deformed Lorentz boost with motion direction along $\widehat{x^i}$.

5.2 Decomposition of the parametric 4-tensor $\delta\omega_{\mu\nu}(g)$ in DSR4

We can now specialize Eq. (44) (expressing the infinitesimal variation of the contravariant 4-vector x^μ in the self-representation) to the DSR4 case by using Eq.s (61) and (62) (ESC on throughout):

$$\delta x^\mu_{(g),DSR4}(\{x\}_{m.}, x^5) = \delta\omega^\mu{}_{\nu,DSR4}(g, x^5)x^\nu =$$

$$= \frac{1}{2}\delta\omega_{\alpha\beta,(DSR4)}(g)(I^{\alpha\beta})^\mu{}_{\nu,DSR4}(x^5)x^\nu =$$

$$= \frac{1}{2}\epsilon^{ij}{}_l\delta\omega_{ij,(DSR4)}(g)(S^l)^\mu{}_{\nu,DSR4}(x^5)x^\nu + \delta\omega_{0i,(DSR4)}(g)(K^i)^\mu{}_{\nu,DSR4}(x^5)x^\nu. \tag{66}$$

Therefore, the parametric 4-tensor $\delta\omega_{\alpha\beta}(g)$ can be written as

$$\delta\omega_{\alpha\beta}(g) = \begin{pmatrix} 0 & -\zeta^1(g) & -\zeta^2(g) & -\zeta^3(g) \\ \zeta^1(g) & 0 & -\theta^3(g) & \theta^2(g) \\ \zeta^2(g) & \theta^3(g) & 0 & -\theta^1(g) \\ \zeta^3(g) & -\theta^2(g) & \theta^1(g) & 0 \end{pmatrix}, \tag{67}$$

where the parameter axial 3-vector $\underline{\theta}(g)$ and the parameter polar 3-vector $\underline{\zeta}(g)$ are respectively defined by:

$$\underline{\theta}(g) = (\theta_i(g)) \equiv \left(-\frac{1}{2}\epsilon_i{}^{jk}\delta\omega_{jk}(g)\right) = (-\delta\omega_{23}(g), -\delta\omega_{31}(g), -\delta\omega_{12}(g)), \tag{68}$$

$$\underline{\zeta}(g) = \zeta_i(g) \equiv (-\delta\omega_{0i}(g)) = (-\delta\omega_{01}(g), -\delta\omega_{02}(g), -\delta\omega_{03}(g)). \tag{69}$$

and correspond to a true deformed rotation and to a deformed boost, respectively.

5.3 The infinitesimal transformations of the 4-d., deformed homogeneous Lorentz group $SO(3,1)_{DEF.}$.

We can now use the results of the previous two Subsections to write Eq. (66) as:

$$\delta x^{\mu}_{(g),DSR4}(\{x\}_{m.}, x^5) = -\theta_l(g)(S^l)^{\mu}{}_{\nu,DSR4}(x^5)x^{\nu} - \zeta_i(g)(K^i)^{\mu}{}_{\nu,DSR4}(x^5)x^{\nu} =$$

$$= \left(-\underline{\theta}(g)\cdot\underline{S_{DSR4}}(x^5) - \underline{\zeta}(g)\cdot\underline{K_{DSR4}}(x^5)\right)^{\mu}{}_{\nu} x^{\nu}. \tag{70}$$

Therefore, the infinitesimal space-time rotation transformation in the deformed Minkowski space $\widetilde{M_4}(x^5)$ corresponding to the element g of $SO(3,1)_{DEF.}$, can be expressed as:

$$\delta g : x^{\mu} \rightarrow (x')^{\mu}_{(g),DSR4}(\{x\}_{m.}, x^5) =$$

$$= (x^{\mu})'_{(g),DSR4}(\{x\}_{m.}, x^5) = x^{\mu} + \delta x^{\mu}_{(g),DSR4}(\{x\}_{m.}, x^5) =$$

$$= (1 - \theta_1(g)S^1_{DSR4}(x^5) - \theta_2(g)S^2_{DSR4}(x^5) - \theta_3(g)S^3_{DSR4}(x^5) +$$

$$-\zeta_1(g)K^1_{DSR4}(x^5) - \zeta_2(g)K^2_{DSR4}(x^5) - \zeta_3(g)K^3_{DSR4}(x^5))^{\mu}{}_{\nu}x^{\nu}, \tag{71}$$

where 1 is the identity of $SO(3,1)_{DEF.}$.

Then, on account of the physical meaning of the 3-d. parameter and generator vectors (respectively $\underline{\theta}(g)$, $\underline{\zeta}(g)$, and $\underline{S_{DSR4}}(x^5)$, $\underline{K_{DSR4}}(x^5)$), we can get, from the matrix representation of the $SO(3,1)_{DEF.}$ generators, the explicit expressions of all the different kinds of infinitesimal transformations of the deformed Lorentz group, namely:

1 - *3-d. deformed space (true) rotations* (parameters $\underline{\theta}(g)$ and generators $\underline{S_{DSR4}}(x^5)$), which constitute the group $SO(3)_{DEF.}$ of rotations in a deformed 3-d. space, non-Abelian, non-invariant proper subgroup of $SO(3,1)_{DEF.}$:

1.1 - (Clockwise) infinitesimal rotation by an angle $\theta_1(g)$ around $\widehat{x^1}$:

$$(x')^{\mu}_{(g),DSR4}(\{x\}_{m.}, x^5) = (x^{\mu})'_{(g),DSR4}(\{x\}_{m.}, x^5) = (1 - \theta_1(g) S^1_{DSR4})^{\mu}{}_{\nu}(x^5) x^{\nu} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} x^{0\prime}_{(g),DSR4}(x^5) \\ x^{1\prime}_{(g),DSR4}(x^5) \\ x^{2\prime}_{(g),DSR4}(x^5) \\ x^{3\prime}_{(g),DSR4}(x^5) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \theta_1(g) b_2^{-2}(x^5) \\ 0 & 0 & -\theta_1(g) b_3^{-2}(x^5) & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} =$$

$$= \begin{pmatrix} x^0 \\ x^1 \\ x^2 + \theta_1(g) b_2^{-2}(x^5) x^3 \\ -\theta_1(g) b_3^{-2}(x^5) x^2 + x^3 \end{pmatrix}; \tag{72}$$

1.2 - (Clockwise) infinitesimal rotation by an angle $\theta_2(g)$ around $\widehat{x^2}$:

$$(x')^{\mu}_{(g),DSR4}(\{x\}_{m.}, x^5) = (x^{\mu})'_{(g),DSR4}(\{x\}_{m.}, x^5) = (1 - \theta_2(g) S^2_{DSR4})^{\mu}{}_{\nu}(x^5) x^{\nu} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} x^{0\prime}_{(g),DSR4}(x^5) \\ x^{1\prime}_{(g),DSR4}(x^5) \\ x^{2\prime}_{(g),DSR4}(x^5) \\ x^{3\prime}_{(g),DSR4}(x^5) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\theta_2(g) b_1^{-2}(x^5) \\ 0 & 0 & 1 & 0 \\ 0 & \theta_2(g) b_3^{-2}(x^5) & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} =$$

$$= \begin{pmatrix} x^0 \\ x^1 - \theta_2(g) b_1^{-2}(x^5) x^3 \\ x^2 \\ \theta_2(g) b_3^{-2}(x^5) x^1 + x^3 \end{pmatrix}; \tag{73}$$

1.3 - (Clockwise) infinitesimal rotation by an angle $\theta_3(g)$ around $\widehat{x^3}$:

$$(x')^{\mu}_{(g),DSR4}(\{x\}_{m.},x^5) = (x^{\mu})'_{(g),DSR4}(\{x\}_{m.},x^5) = (1-\theta_3(g)S^3_{DSR4})^{\mu}{}_{\nu}(x^5)x^{\nu} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} x^{0\prime}_{(g),DSR4}(x^5) \\ x^{1\prime}_{(g),DSR4}(x^5) \\ x^{2\prime}_{(g),DSR4}(x^5) \\ x^{3\prime}_{(g),DSR4}(x^5) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta_3(g)b_1^{-2}(x^5) & 0 \\ 0 & -\theta_3(g)b_2^{-2}(x^5) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} =$$

$$= \begin{pmatrix} x^0 \\ x^1 + \theta_3(g)b_1^{-2}(x^5)x^2 \\ -\theta_3(g)b_2^{-2}(x^5)x^1 + x^2 \\ x^3 \end{pmatrix}. \tag{74}$$

2 - *3-d. deformed space-time (pseudo) rotations, or deformed Lorentz boosts* (parameters $\underline{\zeta}(g)$ and generators $\underline{K_{DSR4}}(x^5)$); they do *not* form a group (see Eq. (82) below):

2.1 - Infinitesimal boost with rapidity $\zeta_1(g)$ along $\widehat{x^1}$:

$$(x')^{\mu}_{(g),DSR4}(\{x\}_{m.},x^5) = (x^{\mu})'_{(g),DSR4}(\{x\}_{m.},x^5) = (1-\zeta_1(g)K^1_{DSR4})^{\mu}{}_{\nu}(x^5)x^{\nu} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} x^{0\prime}_{(g),DSR4}(x^5) \\ x^{1\prime}_{(g),DSR4}(x^5) \\ x^{2\prime}_{(g),DSR4}(x^5) \\ x^{3\prime}_{(g),DSR4}(x^5) \end{pmatrix} = \begin{pmatrix} 1 & -\zeta_1(g)b_0^{-2}(x^5) & 0 & 0 \\ -\zeta_1(g)b_1^{-2}(x^5) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} =$$

$$= \begin{pmatrix} x^0 - \zeta_1(g)b_0^{-2}(x^5)x^1 \\ -\zeta_1(g)b_1^{-2}(x^5)x^0 + x^1 \\ x^2 \\ x^3 \end{pmatrix}; \tag{75}$$

2.2 - Infinitesimal boost with rapidity $\zeta_2(g)$ along $\widehat{x^2}$:

$$(x')^{\mu}_{(g),DSR4}(\{x\}_{m.},x^5)=(x^{\mu})'_{(g),DSR4}(\{x\}_{m.},x^5)=(1-\zeta_2(g)K^2_{DSR4})^{\mu}{}_{\nu}(x^5)x^{\nu}\Leftrightarrow$$

$$\Leftrightarrow\left(\begin{array}{c} x^{0\prime}_{(g),DSR4}(x^5)\\ x^{1\prime}_{(g),DSR4}(x^5)\\ x^{2\prime}_{(g),DSR4}(x^5)\\ x^{3\prime}_{(g),DSR4}(x^5)\end{array}\right)=\left(\begin{array}{cccc} 1 & 0 & -\zeta_2(g)b_0^{-2}(x^5) & 0\\ 0 & 1 & 0 & 0\\ -\zeta_2(g)b_2^{-2}(x^5) & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right)\left(\begin{array}{c} x^0\\ x^1\\ x^2\\ x^3\end{array}\right)=$$

$$=\left(\begin{array}{c} x^0-\zeta_2(g)b_0^{-2}(x^5)x^2\\ x^1\\ -\zeta_2(g)b_2^{-2}(x^5)x^0+x^2\\ x^3\end{array}\right);\tag{76}$$

2.3 - Infinitesimal boost with rapidity $\zeta_3(g)$ along $\widehat{x^3}$:

$$(x')^{\mu}_{(g),DSR4}(\{x\}_{m.},x^5)=(x^{\mu})'_{(g),DSR4}(\{x\}_{m.},x^5)=(1-\zeta_3(g)K^3_{DSR4})^{\mu}{}_{\nu}(x^5)x^{\nu}\Leftrightarrow$$

$$\Leftrightarrow\left(\begin{array}{c} x^{0\prime}_{(g),DSR4}(x^5)\\ x^{1\prime}_{(g),DSR4}(x^5)\\ x^{2\prime}_{(g),DSR4}(x^5)\\ x^{3\prime}_{(g),DSR4}(x^5)\end{array}\right)=\left(\begin{array}{cccc} 1 & 0 & 0 & -\zeta_3(g)b_0^{-2}(x^5)\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\zeta_3(g)b_3^{-2}(x^5) & 0 & 0 & 1\end{array}\right)\left(\begin{array}{c} x^0\\ x^1\\ x^2\\ x^3\end{array}\right)=$$

$$=\left(\begin{array}{c} x^0-\zeta_3(g)b_0^{-2}(x^5)x^3\\ x^1\\ x^2\\ -\zeta_3(g)b_3^{-2}(x^5)x^0+x^3\end{array}\right).\tag{77}$$

The explicit form of the infinitesimal contravariant 4-vector $\delta x^{\mu}_{(g),DSR4}(\{x\}_{m.},x^5)$,

corresponding to an element $g \in SO(3,1)_{DEF.}$, is therefore:

$$
\begin{cases}
\delta x^0_{(g),DSR4}(\{x\}_{m.}, x^5) = -\zeta_1(g) b_0^{-2}(x^5) x^1 - \zeta_2(g) b_0^{-2}(x^5) x^2 - \zeta_3(g) b_0^{-2}(x^5) x^3 = \\
\qquad = b_0^{-2}(x^5)(-\zeta_1(g)x^1 - \zeta_2(g)x^2 - \zeta_3(g)x^3), \\
\\
\delta x^1_{(g),DSR4}(\{x\}_{m.}, x^5) = -\zeta_1(g) b_1^{-2}(x^5) x^0 + \theta_3(g) b_1^{-2}(x^5) x^2 - \theta_2(g) b_1^{-2}(x^5) x^3 = \\
\qquad = -b_1^{-2}(x^5)(\zeta_1(g)x^0 - \theta_3(g)x^2 + \theta_2(g)x^3), \\
\\
\delta x^2_{(g),DSR4}(\{x\}_{m.}, x^5) = -\zeta_2(g) b_2^{-2}(x^5) x^0 - \theta_3(g) b_2^{-2}(x^5) x^1 + \theta_1(g) b_2^{-2}(x^5) x^3 = \\
\qquad = -b_2^{-2}(x^5)(\zeta_2(g)x^0 + \theta_3(g)x^1 - \theta_1(g)x^3), \\
\\
\delta x^3_{(g),DSR4}(\{x\}_{m.}, x^5) = -\zeta_3(g) b_3^{-2}(x^5) x^0 + \theta_2(g) b_3^{-2}(x^5) x^1 - \theta_1(g) b_3^{-2}(x^5) x^2 = \\
\qquad = -b_3^{-2}(x^5)(\zeta_3(g)x^0 - \theta_2(g)x^1 + \theta_1(g)x^2).
\end{cases}
\tag{78}
$$

The covariant components of such a 4-vector are

$$
\begin{cases}
\delta x_{0(g),DSR4}(\{x\}_{m.}) = -\zeta_1(g)x^1 - \zeta_2(g)x^2 - \zeta_3(g)x^3, \\
\\
\delta x_{1(g),DSR4}(\{x\}_{m.}) = \zeta_1(g)x^0 - \theta_3(g)x^2 + \theta_2(g)x^3, \\
\\
\delta x_{2(g),DSR4}(\{x\}_{m.}) = \zeta_2(g)x^0 + \theta_3(g)x^1 - \theta_1(g)x^3, \\
\\
\delta x_{3(g),DSR4}(\{x\}_{m.}) = \zeta_3(g)x^0 - \theta_2(g)x^1 + \theta_1(g)x^2.
\end{cases}
\tag{79}
$$

Comparing this last result with the expression (25) of the covariant Killing vector, we see the perfect matching between the space-time rotational component of $\xi_\mu(\{x\}_{m.})$ (unique for all the 4-d. generalized Minkowski spaces) and the covariant 4-vector $\delta x_{\mu(g),DSR4}(\{x\}_{m.})$ related to $SO(3,1)_{DEF.}$.

5.4 The 4-d. deformed Lorentz algebra, i.e. the Lie algebra of the 4-d. deformed, homogeneous Lorentz group $SO(3,1)_{DEF.}$.

Let us specialize Eq. (34) to the DSR4 case, in order to derive the 4-d. deformed Lorentz algebra, i.e. the Lie algebra of the 4-d. deformed, homogeneous Lorentz group $SO(3,1)^{6par.}_{DEF.}$. We get:

$$[I^{\alpha\beta}_{DSR4}(x^5), I^{\rho\sigma}_{DSR4}(x^5)] =$$

$$= g^{\alpha\sigma}_{DSR4}(x^5) I^{\beta\rho}_{DSR4}(x^5) + g^{\beta\rho}_{DSR4}(x^5) I^{\alpha\sigma}_{DSR4}(x^5) +$$

$$-g^{\alpha\rho}_{DSR4}(x^5) I^{\beta\sigma}_{DSR4}(x^5) - g^{\beta\sigma}_{DSR4}(x^5) I^{\alpha\rho}_{DSR4}(x^5) =$$

$$= \delta^{\alpha\sigma}\left(b_0^{-2}(x^5)\delta^{\alpha 0} - b_1^{-2}(x^5)\delta^{\alpha 1} - b_2^{-2}(x^5)\delta^{\alpha 2} - b_3^{-2}(x^5)\delta^{\alpha 3}\right) I^{\beta\rho}_{DSR4}(x^5) +$$

$$+\delta^{\beta\rho}\left(\delta^{\beta 0}b_0^{-2}(x^5) - \delta^{\beta 1}b_1^{-2}(x^5) - \delta^{\beta 2}b_2^{-2}(x^5) - \delta^{\beta 3}b_3^{-2}(x^5)\right) I^{\alpha\sigma}_{DSR4}(x^5) +$$

$$-\delta^{\alpha\rho}\left(\delta^{\alpha 0}b_0^{-2}(x^5) - \delta^{\alpha 1}b_1^{-2}(x^5) - \delta^{\alpha 2}b_2^{-2}(x^5) - \delta^{\alpha 3}b_3^{-2}(x^5)\right) I^{\beta\sigma}_{DSR4}(x^5) +$$

$$-\delta^{\beta\sigma}\left(\delta^{\beta 0}b_0^{-2}(x^5) - \delta^{\beta 1}b_1^{-2}(x^5) - \delta^{\beta 2}b_2^{-2}(x^5) - \delta^{\beta 3}b_3^{-2}(x^5)\right) I^{\alpha\rho}_{DSR4}(x^5). \tag{80}$$

On account of the physical interpretation of the infinitesimal generators, one has therefore the following kinds of commutation relations:

1 - Commutator of generators of 3-d. deformed space rotations:

$$[I^{ij}_{DSR4}(x^5), I^{lm}_{DSR4}(x^5)] =$$

$$= \delta^{im}\left(\delta^{i0}b_0^{-2}(x^5) - \delta^{i1}b_1^{-2}(x^5) - \delta^{i2}b_2^{-2}(x^5) - \delta^{i3}b_3^{-2}(x^5)\right) I^{jl}_{DSR4}(x^5) +$$

$$+\delta^{jl}\left(\delta^{j0}b_0^{-2}(x^5) - \delta^{j1}b_1^{-2}(x^5) - \delta^{j2}b_2^{-2}(x^5) - \delta^{j3}b_3^{-2}(x^5)\right) I^{im}_{DSR4}(x^5) +$$

$$-\delta^{il}\left(\delta^{i0}b_0^{-2}(x^5) - \delta^{i1}b_1^{-2}(x^5) - \delta^{i2}b_2^{-2}(x^5) - \delta^{i3}b_3^{-2}(x^5)\right) I^{jm}_{DSR4}(x^5) +$$

$$-\delta^{jm}\left(\delta^{j0}b_0^{-2}(x^5) - \delta^{j1}b_1^{-2}(x^5) - \delta^{j2}b_2^{-2}(x^5) - \delta^{j3}b_3^{-2}(x^5)\right) I^{il}_{DSR4}(x^5) =$$

$$\overset{\text{ESC off on } i\ ,\ j}{=} -\delta^{im}b_i^{-2}(x^5)I^{jl}_{DSR4}(x^5) - \delta^{jl}b_j^{-2}(x^5)I^{im}_{DSR4}(x^5) +$$

$$+\delta^{il}b_i^{-2}(x^5)I^{jm}_{DSR4}(x^5) + \delta^{jm}b_j^{-2}(x^5)I^{il}_{DSR4}(x^5); \qquad (81)$$

2 - Commutator of generators of 3-d. deformed boosts:

$$[I^{i0}_{DSR4}(x^5), I^{j0}_{DSR4}(x^5)] =$$

$$= \delta^{i0}\left(\delta^{i0}b_0^{-2}(x^5) - \delta^{i1}b_1^{-2}(x^5) - \delta^{i2}b_2^{-2}(x^5) - \delta^{i3}b_3^{-2}(x^5)\right) I^{0j}_{DSR4}(x^5) +$$

$$+\delta^{0j}\left(\delta^{j0}b_0^{-2}(x^5) - \delta^{j1}b_1^{-2}(x^5) - \delta^{j2}b_2^{-2}(x^5) - \delta^{j3}b_3^{-2}(x^5)\right) I^{i0}_{DSR4}(x^5) +$$

$$-\delta^{ij}\left(\delta^{i0}b_0^{-2}(x^5) - \delta^{i1}b_1^{-2}(x^5) - \delta^{i2}b_2^{-2}(x^5) - \delta^{i3}b_3^{-2}(x^5)\right) I^{00}_{DSR4}(x^5) +$$

$$-\delta^{00}\left(\delta^{00}b_0^{-2}(x^5) - \delta^{01}b_1^{-2}(x^5) - \delta^{02}b_2^{-2}(x^5) - \delta^{03}b_3^{-2}(x^5)\right) I^{ij}_{DSR4}(x^5) =$$

$$= -b_0^{-2}(x^5) I^{ij}_{DSR4}(x^5); \qquad (82)$$

3 - "Mixed" commutator of 3-d. deformed space and boost generators:

$$[I^{ij}_{DSR4}(x^5), I^{k0}_{DSR4}(x^5)] =$$

$$= \delta^{i0}\left(\delta^{i0}b_0^{-2}(x^5) - \delta^{i1}b_1^{-2}(x^5) - \delta^{i2}b_2^{-2}(x^5) - \delta^{i3}b_3^{-2}(x^5)\right) I^{jk}_{DSR4}(x^5) +$$

$$+\delta^{jk}\left(\delta^{j0}b_0^{-2}(x^5) - \delta^{j1}b_1^{-2}(x^5) - \delta^{j2}b_2^{-2}(x^5) - \delta^{j3}b_3^{-2}(x^5)\right) I^{i0}_{DSR4}(x^5) +$$

$$-\delta^{ik}\left(\delta^{i0}b_0^{-2}(x^5) - \delta^{i1}b_1^{-2}(x^5) - \delta^{i2}b_2^{-2}(x^5) - \delta^{i3}b_3^{-2}(x^5)\right) I^{j0}_{DSR4}(x^5) +$$

$$-\delta^{j0}\left(\delta^{j0}b_0^{-2}(x^5) - \delta^{j1}b_1^{-2}(x^5) - \delta^{j2}b_2^{-2}(x^5) - \delta^{j3}b_3^{-2}(x^5)\right) I^{ik}_{DSR4}(x^5) =$$

$$\overset{\text{ESC off on } i \text{ and } j\text{, as before}}{=} -\delta^{jk}b_j^{-2}(x^5)I^{i0}_{DSR4}(x^5) + \delta^{ik}b_i^{-2}(x^5)I^{j0}_{DSR4}(x^5). \tag{83}$$

In the "self-representation" basis of $SO(3,1)^{6}_{DEF.}$, it is easy to show that

commutation relations (81)-(83) read[10]:

$$\begin{cases} i)\ \ [S^i_{DSR4}(x^5), S^j_{DSR4}(x^5)] \overset{\text{ESC on}}{=} \\ \\ = \left(\sum_{s=1}^3 (1-\delta_{is})((1-\delta_{js})b_s^{-2}(x^5)\right)\epsilon_{ijk}S^k_{DSR4}(x^5) = \epsilon_{ijk}b_k^{-2}(x^5)S^k_{DSR4}(x^5), \\ \\ \\ ii)\ \ [K^i_{DSR4}(x^5), K^j_{DSR4}(x^5)] \overset{\text{ESC on}}{=} -b_0^{-2}(x^5)\epsilon_{ijk}S^k_{DSR4}(x^5), \\ \\ \\ iii)\ \ [S^i_{DSR4}(x^5), K^j_{DSR4}(x^5)] \overset{\text{ESC on on } l, \text{ ESC off on } j}{=} \\ \\ = \epsilon_{ijl}K^l_{DSR4}(x^5)\left(\sum_{s=1}^3 \delta_{js}b_s^{-2}(x^5)\right) = \epsilon_{ijl}b_j^{-2}(x^5)K^l_{DSR4}(x^5), \end{cases} \tag{84}$$

which define the deformed Lorentz algebra of generators of $SO(3,1)^{6}_{DEF.}$.

Such relations generalize to the DSR4 case the infinitesimal algebraic structure of the standard homogeneous Lorentz group $SO(3,1)$. They admit interpretations completely analogous to those of the usual Lorentz algebra:

i) of Eq. (84) expresses the closed nature of the algebra of the deformed rotation generators; consequently the 3-d. deformed space rotations form a 3-parameter subgroup of $SO(3,1)^{6}_{DEF.}$, namely $SO(3)_{DEF.}$;

on the contrary, the deformed boost generator algebra is not closed (according to *ii*) of Eq. (84), and then the deformed boosts do *not* form a

[10]Use has been made of the relation

$$\epsilon_{ims}\epsilon_{jrs}b_s^{-2}(x^5) =$$

$$= (\delta_{ij}\delta_{mr} - \delta_{ir}\delta_{mj})\left(\sum_{k=1}^{3}(1-\delta_{ik})(1-\delta_{mk})b_k^{-2}(x^5)\right),$$

which generalizes to the DSR4 case the well-known formula $\epsilon_{ims}\epsilon_{jrs} = \delta_{ij}\delta_{mr} - \delta_{ir}\delta_{jm}$.

subgroup of the deformed Lorentz group. This implies that $SO(3,1)^{6}_{DEF.}$ *cannot* be considered the product of two its subgroups;

this is further confirmed by the non-commutativity of deformed space rotations and boosts, expressed by *iii*) of Eq. (84).

Moreover, *i*) and *iii*) of Eq. (84) show that both $\underline{S_{DSR4}}(x^5)$ and $\underline{K_{DSR4}}(x^5)$ behave as 3-vectors under deformed spatial rotations.

6 Conclusions

We want first to stress that, in the 4-dimensional case with the hyperbolic metric signature ($S = 3, T = 1$), in the limit

$$g_{\mu\nu,DSR4}(x^5) \to g_{\mu\nu,SSR4} \Leftrightarrow$$

$$\Leftrightarrow \delta_{\mu\nu}(\delta_{\mu 0} b_0^2(x^5) - \delta_{\mu 1} b_1^2(x^5) - \delta_{\mu 2} b_2^2(x^5) - \delta_{\mu 3} b_3^2(x^5)) \to$$

$$\to \delta_{\mu\nu}(\delta_{\mu 0} - \delta_{\mu 1} - \delta_{\mu 2} - \delta_{\mu 3}) \Leftrightarrow b_\mu^2(x^5) \to 1, \quad \forall \mu = 0,1,2,3 \qquad (85)$$

all results valid at group-transfomation level in DSR4 reduce to the standard ones in SR.

Moreover, let us recall that the (parametric) dependence of the metric of a generalized Minkowski space on the set $\{x\}_{n.m.}$ of non-metrical coordinates reflects itself also at the group level. In particular, such a dependence shows up in:

1) the $N \times N$ matrix representation of the infinitesimal generators;

2) the infinitesimal group transformations;

3) the structure constants of the Lie algebra of generators .

Let us also notice that, since to *any* fixed value $\{\bar{x}\}_{n.m.}$ of $\{x\}_{n.m.}$ there corresponds a generalized Minkowski space $\widetilde{M_N}(\{\bar{x}\}_{n.m.})$, we have a family of N-d. generalized Minkowski spaces

$$\left\{\widetilde{M_N}(\{x\}_{n.m.})\right\}_{\{x\}_{n.m.} \in R_{\{x\}_{n.m.}}}, \qquad (86)$$

where $R_{\{x\}_{n.m.}}$ is the range of the set $\{x\}_{n.m.}$; if the cardinality of the range of each element of the set $\{x\}_{n.m.}$ is infinite, the cardinality of $R_{\{x\}_{n.m.}}$ (and of

the family (86)) is $\infty^{N_{n.m.}}$. In correspondence, one gets a family of generalized Poincaré groups

$$\left\{P(S,T)^{N(N+1)/2}_{GEN.}(\{x\}_{n.m.})\right\}_{\{x\}_{n.m.}\in R_{\{x\}_{n.m.}}}, \tag{87}$$

with the same cardinality structure as (86).

This can be summarized in the following scheme:

$$\left.\begin{array}{c}
\textit{(Iper)spatial level} \text{ of } N\text{-d. generalized Minkowski spaces:} \\
1) \left\{\widetilde{M_N}(\{x\}_{n.m.})\right\}_{\{x\}_{n.m.}\in R_{\{x\}_{n.m.}}} \\
\\
2)\ \widetilde{M_N}(\{x\}_{n.m.}) \equiv \widetilde{M_N}(\{\bar{x}\}_{n.m.})
\end{array}\right\} \Leftrightarrow$$

$$\Leftrightarrow \left\{\begin{array}{c}
\textit{Group level} \text{ of related maximal Killing groups:} \\
1) \left\{P(S,T)^{N(N+1)/2}_{GEN.}(\{x\}_{n.m.})\right\}_{\{x\}_{n.m.}\in R_{\{x\}_{n.m.}}} = \\
= \left\{SO(T,S)^{N(N-1)/2}_{GEN.}(\{x\}_{n.m.}) \otimes_s Tr.(T,S)^{N}_{GEN.}(\{x\}_{n.m.})\right\}_{\{x\}_{n.m.}\in R_{\{x\}_{n.m.}}} \\
\\
2)\ P(S,T)^{N(N+1)/2}_{GEN.}(\{x\}_{n.m.}) \equiv P(S,T)^{N(N+1)/2}_{GEN.}(\{\bar{x}\}_{n.m.}).
\end{array}\right. \tag{88}$$

In the forthcoming papers, we will discuss the finite structure of the space-time rotation groups, and the translation component of the maximal Killing group of generalized Minkowski spaces.

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